**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part I. Variations calculus

We consider different problems of minimization integral functionals. Differential equations are obtained as necessary conditions of the extremum. The theory is illustrated by physical examples.

## Lecture 3. Euler equation for Lagrange problem

The problem of integral functional minimization with given boundary condition is considered. The second order differential Euler equation is obtained as necessary condition of the optimality. The minimization of length between points, the fall of a body, the refraction of the light, and Brachistochrone problem are considered as examples.

### 3.1. Lagrange problem

Consider the functional



where  *F* is a given sufficiently regular function,  is unknown function, which satisfies boundary conditions

 (3.1)

 and  are given numbers.

**Problem 3.1**. *Find the function v*, *which minimizes the functional I and satisfies the boundary conditions* (3.1).

**Definition 3.1**. *Problem* 3.1 *is called* ***Lagrange problem****.*

### 3.2. Euler equation

Let the function *u* be the solution of Lagrange problem. Definite the function



where *σ* is a number, *h* is smooth enough function on the interval , which satisfies the homogeneous boundary conditions

 (3.2)

Then  satisfies boundary conditions (3.2) (see Figure 3.1).

**Definition 3.2.** *The function  is called* ***the******variation of the function*** *u*.



Figure 3.1. Variation of the function.

The functional *I* has the minimum at the point *u* if and only if the number 0 is the point of the minimum for the function *f*.

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| **Conclusion**: *The problem of the functional minimization is transformed  to the problem of the minimization for the function of one variable.* |

By stationary condition the necessary condition of the minimum of the differentiable function is the equality to zero of its derivative. Let the function *F* be differentiable. Denote by  and  the partial derivatives of the function with respect to second and third arguments.

**Lemma 3.1**. *The derivative of the function f at zero is equal to*

 (3.3)

**Proof**. Define



Using Taylor formula we get



where  as  Then



Divide by *σ* and pass to the limit as . We have



Using (3.2) we get



So the equality (3.3) is true.

**Definition 3.3.** *The derivative of the function f at zero is called* ***the variation of the functional*** *I at the point u and denote by* *or* 

Using stationary condition we obtain the equality

 (3.4)

which is true for all function *h,* which satisfies the boundary conditions (1.6). So we have

**Theorem 3.1**. *The variation of the functional is equal to zero in the point of the minimum.*

We transform the equality (3.4) with using the following result.

**Lemma 3.2 (Basic lemma in the Calculus of variations)**. *Let g be a continuous function on the interval*  *and satisfies the equality*

 (3.5)

*for all continuous function h*. *Then the function g* *is equal to zero everywhere.*

Using (3.4), (3.5) we get

 (3.6)

By Lemma 3.2 we obtain

**Theorem 3.2**. *Let the smooth enough function u be a solution of Lagrange problem, then it satisfies* ***Euler equation***

 (3.7)

*in* .

**Definition 3.3.** *The solution of Euler equation is called* ***extremal****.*

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| **Question**: *What kind of equations has Euler equation?* |

Euler equation is a second order ordinary differential equation.

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| **Question**: *The solution of the second order differential equation depends from two arbitrary constants. How we can determine it?* |

We add the boundary conditions (3.2) for solving Euler equation.

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| **Conclusion**: *Lagrange Problem is transformed to the boundary problem* (3.1), (3.7). |

The algorithm of solving of Lagrange Problem is given in Table 3.1.

Table. 3.1. The algorithm of solving of Lagrange Problem

|  |  |  |
| --- | --- | --- |
| **step** | **action** | **remark** |
| 1 | Definition of the concrete values of *F*, , . | The concrete problem is transformed to the standard form. |
| 2 | Definition of Euler Equation. | Calculation of the partial derivatives of *F* and definition of the concrete equation (3.7). |
| 3 | Finding of the general solution of Euler Equation. | Finding the general solution of the equation (3.7), that depends from two constants. |
| 4 | Using the boundary conditions. | Definition of the unknown constants from the equalities (3.1). |
| 5 | Calculation the corresponding value of the functional. | Definition of the corresponding value of *I*. |
| 6 | Analysis of the result. | This result can be no minimum  of the functional. |

### 3.3. Examples

**Example 3.1**. *Minimize the functional*



*with boundary conditions*



We will analyze the given problem with using of the known algorithm (see Table 3.1).

Determine the parameters of the concrete problem:



Determine Euler equation. We have



So Euler equation transform to



This equation with given boundary conditions has the solution  It can be the solution of our problem. In really our functional can be transform to



Its value is greater or equal -1, besides the value -1 can be realized only for  So it is the solution of our problem.

**Example 3.2.** Minimize the functional

 (3.8)

with boundary conditions

 (3.9)

We have Lagrange Problem with function

.

Determine Euler equation



We obtain

.

Then



so we have



Hence the derivative  equal to a constant . So we have  then



We use equality (3.8) for finding two unknown constants. We get



Then we find



Note that the functional (3.8) determine the distance by the curve  between the points determine by equalities (3.9). Our result is trivial because the corresponding curve is the line.

**Remark 3.1**. The problems of maximization of the corresponding functional have the same Euler equations with the same boundary conditions. However the found solutions minimize the considered functionals. So it is not solutions of the maximizations problems.

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| **Conclusion**: *Euler equation is a necessary condition  of extremum. It can be no sufficient condition.* |

**Remark 3.2**. We determined Euler equation with using the stationary condition . It has a necessary condition of minimum only for the arbitrary function *f*. So Euler equation is a necessary 0

### 3.4. The fall of the body

We consider the fall of the body. This phenomenon is described by the height *y.* We determine the mechanical energy of the body *E*(*t*) in the arbitrary time *t*. It is the sum of the kinetic energy *K*(*t*) and the potential *U*(*t*). Then



The potential energy is the product between the weight *P* and the height *y*

*U*(*t*) = -*Р* *y*(*t*) = -*mgу*(*t*),

where *m* is the mass of the body, and *g* is the gravitational acceleration. We use the sign “minus” here because the direction of the force is the contrary of the increasing of the height.

The kinetic energy is proportional to the square of the velocity



Then the mechanical energy of the body in the time *t* is equal to

 (3.10)

If the energy has the constant value *Е*\*, then the energy of the body from the initial time *t*0 to the final time *t*1 is equal to *Е*\*(*t*1 – *t*0). In really the energy is variable. So we can determine the value



with is called the ***action of the system*** on the time interval [*t*0,*t*1]. Let the initial and final height of the body are known. We have boundary conditions

*y*(*t*0) = *y*0, *y*(*t*1) = *y*1. (3.11)

We have the problem of the minimization of the value *I* on the set of the functions *y*, which satisfy the boundary conditions (3.11).

The function *F* is equal



for this case. So we obtain Euler equation



Hence we get the known equation of the fall of the body



### Task 2. Euler equation for Lagrange problem

* Euler equation is the second order ordinary differential equation.
* Euler equation is solved with given boundary conditions.
* Lagrange problem has applications in the geometry, mechanics, and optics.

Find the function  which minimizes the functional



and satisfies the boundary conditions



The values of the parameters.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **variant** |  |  |  |  |  | **name** |
| 1 |  | 0 |  | 0 | 1 |  |
| 2 |  | 0 | 1 | 0 | 1 |  |
| 3 |  | 0 |  | 0 | -1 |  |
| 4 |  | 0 | 1 | 0 | 1 |  |
| 5 |  | 0 |  | 0 | 1 |  |
| 6 |  | 0 | 1 | 0 | 1 |  |
| 7 |  | 0 | 1 | 0 | 1 |  |
| 8 |  | 0 |  | 1 | 0 |  |
| 9 |  | 0 | 1 | 0 | 1 |  |
| 10 |  | 0 | *π* | 0 | 1 |  |
| 11 |  | 0 |  | 0 | 1 |  |
| 12 |  | 0 |  | 0 | 1 |  |
| 13 |  | 0 |  | 0 | -1 |  |
| 14 |  | 0 |  | 0 | -1 |  |
| 15 |  | 0 |  | 0 | 1 |  |
| 16 |  | 0 |  | 0 | 1 |  |
| 17 |  | 0 |  | 1 | 0 |  |
| 18 |  | 0 |  | 1 | 0 |  |
| 19 |  | 0 |  | 0 | -1 |  |
| 20 |  | 0 |  | 0 | -1 |  |

It is necessary to make the following actions:

1. Give the problem statement.
2. Determine Euler equation.
3. Find the general solution of Euler equation, which depends from two constants.
4. Find these constants by means of the given boundary conditions.
5. Find the corresponding solution of the boundary problem.
6. Calculate the corresponding value of the given functional.
7. Calculate the value of the given functional for the linear function which satisfies the given boundary conditions.
8. Compare these results.

It will be recalled that the general solution of the differential equation  is

.

The general solution of the differential equation  is 

### 3.5. First integrals

Consider general form of second order differential equation



This equation is difficult enough for the general function *F*. However sometimes we can transform it to easier.

**Definition 3.3.** *The function*  *is called* ***the first integral*** *of this system*, *if G is constant for all solution of the given equation.*

If we can find the first integral of the system, we transform given second order differential equation to the first order differential equation



The equation (3.12) has the first order. So it is simpler than the standard Euler equation. We consider application of this result. Let us have the function *F*, which does not depend from the independent variable *x.*

**Lemma 3.3**. *Let the function F does not depend from the independent variable x, then the Euler equation can be transformed to the equality*

 (3.12)

*where c is a constant*.

**Proof**. Find

.

The right side of this equality is equal to zero because of Euler equation. So we obtain the equality



Then the equality (3.12) holds.

**Corollary.** *The function*  *is the first integral**of the system*.

The equation (3.12) has the first order. So it is simpler than the standard Euler equation. We consider application of this result.

### 3.5. The fall of the body and its first integral

We considered before the phenomenon of the fall of the body. It was described by the problem

of the minimization of the action



### 3.5. Fermat principle and the refraction of light low

One of the known optical low is Fermat principle. By Fermat principle the light moves by the curve, which gives the minimum of the time of the movement. We use it for determine the low of the refraction of light. We consider the movement of the light on the plane by the curve . The initial and final points are known. So we have the conditions

 (3.13)

The velocity of the movement is determine by the formula



where the function  is the way of the light. Then we have the equality



We know that the way by the curve  can be determine by the equality



Then the previous formula can be transformed to the equality



We know the initial point  at the initial time  and the final point  for the final time  After integration we get

 (3.14)

By Fermat principle the curve  of the light movement minimizes the time (3.14).

Let the point  be the boundary of two different environments (see Figure 3.2). The velocity of the light is constant in each environment. So it is determine by the formula

 (3.15)



Figure 3.2. Refraction of the light.

In our case the function *F* in Euler equation does not depend from *x*. So we can us Lemma 3.3. For our case the equality (3.12) can be transformed to



Then we have

 (3.16)

where

.

The low of movement  can be found from the differential equation (3.16).

We know that the derivative is the tangent of the angle of the curve . Use the known formula



Then the equality (3.16) is transformed to



Using (3.14) we get the equality



This equality gives the relation between the angles of the sight and the refraction and the velocities of the light for different environments. This is light refraction low of Snellius.

### 3.6. Brachistochrone problem

We return to consider *Brachistochrone problem*. We would like to determine the curve *у* = *у*(*х*) from the origin of coordinates to the point with coordinates (*х*1,*у*1) such that the time of movement by influence of the weight only is minimal. We know (see Lecture 1), that the time of this movement can be described by the integral



We have the problem of its minimization with boundary conditions

*у*(0) = 0, *у*(*х*1) = *у*1. (3.17)

Our functional does not depend from *x*. So we can use Lemma 3.3 with



Find



Then we have the equation (3.12)



where *c* is a constant. So we obtain the equality



Therefore we obtain



where  Hence we get the equation



We have the equalities



So we get



where *b* is constant. Make the substitution  in the first integral and  in the second integral. We obtain



Hence we have the general solution of Euler equation



Using first boundary condition (3.17) we get  So we have the equality

 (3.18)

where the constant *a* can be found from the second boundary condition (3.17). It depends from the concrete coordinates if the final point of the movement. The function described by the equation (3.18) is called cycloid.

### Task 3. Euler equation. Special cases

We have Lagrange problem with the functionals





where the functions *F*, *G* and *H* are given. We have also the boundary conditions



It is necessary to choose the parameters  such that Euler equation has a solution.

The values of the parameters.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **variant** |  |  |  | **name** |
| 1 |  |  |  |  |
| 2 |  |  |  |  |
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| 10 |  |  |  |  |

### Outcome

* The problem of the functional minimization can be transformed to the problem of the minimization of the function with one variable by means of the variational method.
* Euler equation is the necessary condition of the minimum for Lagrange problem, notably its solution can minimize the given functional.

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### Next step

The problems of minimization for integral functionals have a lot of practical applications. We know the general method of solving these problems. We will try to extend these results to other problems. At first we will consider problems of functional minimization with many unknown functions.